

REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS WITH COMMUTING RICCI TENSOR

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ABSTRACT. In this paper we first introduce the full expression of the curvature tensor of a real hypersurface M in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 2$ from the equation of Gauss. Next we derive a new formula for the Ricci tensor of M in $SU_{2,m}/S(U_2 \cdot U_m)$. Finally we give a complete classification of Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor. Each can be described as a tube over a totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere whose center at infinity is singular.

INTRODUCTION

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms $Q_m(c)$ Kimura [8] and [9] (resp. Pérez and the author [15]) considered real hypersurfaces in $M_n(c)$ (resp. in $Q_m(c)$) with commuting Ricci tensor, that is, $S\phi = \phi S$, (resp. $S\phi_i = \phi_i S$, $i = 1, 2, 3$) where S and ϕ (resp. S and ϕ_i , $i = 1, 2, 3$) denote the Ricci tensor and the structure tensor of real hypersurfaces in $M_m(c)$ (resp. in $Q_m(c)$).

In [8] and [9], Kimura has classified that a Hopf hypersurface M in complex projective space $P_m(\mathbb{C})$ with commuting Ricci tensor is locally congruent to of type (A), a tube over a totally geodesic $P_k(\mathbb{C})$, of type (B), a tube over a complex quadric Q_{m-1} , $\cot^2 2r = m-2$, of type (C), a tube over $P_1(\mathbb{C}) \times P_{(m-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{m-2}$ and n is odd, of type (D), a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$, of type (E), a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $m = 15$.

On the other hand, in a quaternionic projective space $\mathbb{Q}P^m$ Pérez and the author [15] have classified real hypersurfaces in $\mathbb{Q}P^m$ with commuting Ricci tensor $S\phi_i = \phi_i S$, $i = 1, 2, 3$, where S (resp. ϕ_i) denotes the Ricci tensor (resp. the structure tensor) of M in $\mathbb{Q}P^m$, is locally congruent to of A_1, A_2 -type, that is, a tube over $\mathbb{Q}P^k$ with radius $0 < r < \frac{\pi}{2}$, $k \in \{0, \dots, m-1\}$. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where J_i , $i = 1, 2, 3$, denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and N a unit normal field of M in $\mathbb{Q}P^m$. Moreover, Pérez and Suh [14] have considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$,

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where R denotes the curvature tensor of a real hypersurface M in $\mathbb{Q}P^m$, and proved that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over $\mathbb{Q}P^k$.

Let us denote by $SU_{2,m}$ the set of $(m+2) \times (m+2)$ -indefinite special unitary matrices and U_m the set of $m \times m$ -unitary matrices. Then the Riemannian symmetric space $SU_{2,m}/S(U_2U_m)$, $m \geq 2$, which consists of complex two-dimensional subspaces in indefinite complex Euclidean space \mathbb{C}_2^{m+2} , has a remarkable feature that it is a Hermitian symmetric space as well as a quaternionic Kähler symmetric space. In fact, among all Riemannian symmetric spaces of noncompact type the symmetric spaces $SU_{2,m}/S(U_2U_m)$, $m \geq 2$, are the only ones which are Hermitian symmetric and quaternionic Kähler symmetric.

The existence of these two structures leads to a number of interesting geometric problems on $SU_{2,m}/S(U_2U_m)$, one of which we are going to study in this article. To describe this problem, we denote by J the Kähler structure and by \mathfrak{J} the quaternionic Kähler structure on $SU_{2,m}/S(U_2U_m)$. Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$ and denote by N a unit normal to M . Then a structure vector field ξ defined by $\xi = -JN$ is said to be a Reeb vector field.

Next, we consider the standard embedding of $SU_{2,m-1}$ in $SU_{2,m}$. Then the orbit $SU_{2,m-1} \cdot o$ of $SU_{2,m-1}$ through o is the Riemannian symmetric space $SU_{2,m-1}/S(U_2U_{m-1})$ embedded in $SU_{2,m}/S(U_2U_m)$ as a totally geodesic submanifold. Every tube around $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ has the property that both maximal complex subbundle \mathcal{C} and quaternionic subbundle \mathcal{Q} are invariant under the shape operator.

Finally, let m be even, say $m = 2n$, and consider the standard embedding of $Sp_{1,n}$ in $SU_{2,2n}$. Then the orbit $Sp_{1,n} \cdot o$ of $Sp_{1,n}$ through o is the quaternionic hyperbolic space $\mathbb{H}H^n$ embedded in $SU_{2,2n}/S(U_2U_{2n})$ as a totally geodesic submanifold. Any tube around $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$ has the property that both \mathcal{C} and \mathcal{Q} are invariant under the shape operator.

As a converse of the statements mentioned above, we assert that with one possible exceptional case there are no other such real hypersurfaces. Related to such a result, we introduce another theorem due to Berndt and Suh [3] as follows:

Theorem A. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then the maximal complex subbundle \mathcal{C} of TM and the maximal quaternionic subbundle \mathcal{Q} of TM are both invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces:*

- (A) *a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$;*
 - (B) *a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$;*
 - (C) *a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular;*
- or the following exceptional case holds:*

(D) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), \quad T_\gamma = J(TM \ominus \mathcal{Q}), \quad T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

In Theorem A the maximal complex subbundle \mathcal{C} of TM is invariant under the shape operator if and only if the Reeb vector field ξ becomes a principal vector field for the shape operator A of M in $SU_{2,m}/S(U_2 \cdot U_m)$. In this case the Reeb vector field ξ is said to be a Hopf vector field. The flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow*.

The classification of all real hypersurfaces in complex projective space $\mathbb{C}P^m$ with isometric *Reeb flow* has been obtained by Okumura [12]. The corresponding classification in complex hyperbolic space $\mathbb{C}H^m$ is due to Montiel and Romero [11] and in quaternionic projective space $\mathbb{H}P^m$ due to Martinez and Pérez [10] respectively.

Now let us introduce a classification theorem due to Suh [19] for all real hypersurfaces with isometric *Reeb flow* in complex hyperbolic two-plane Grassmann manifold $SU_{2,m}/S(U_2 \cdot U_m)$ as follows:

Theorem B. *Let M be a connected orientable real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular.*

In the proof of Theorem A we proved that the 1-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution \mathcal{Q}^\perp or in the orthogonal complement \mathcal{Q} such that $T_x M = \mathcal{Q} \oplus \mathcal{Q}^\perp$. The case (A) in Theorem A is just the case that the 1-dimensional distribution $[\xi]$ belongs to the distribution \mathcal{Q} . Of course, it is not difficult to check that the Ricci tensor S of type (A) or of type (C) with $JX \in \mathfrak{J}X$ in Theorem A commutes with the structure tensor, that is $S\phi = \phi S$. Then it must be a natural question to ask whether real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor can exist or not.

In this paper we consider such a converse problem and want to give a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $S\phi = \phi S$ as follows:

Main Theorem. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$ in $SU_{2,m}/S(U_2 \cdot U_m)$ or a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular.*

A remarkable consequence of our Main Theorem is that a connected complete real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$ with commuting Ricci tensor is homogeneous and has an isometric Reeb flow. This was also true in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which could be identified with symmetric space of compact type $SU_{m+2}/S(U_2 \cdot U_m)$, as follows from the classification. It would be interesting to understand the actual reason for it (See [2], [17] and [18]).

This paper is organized as follows. In Section 1 we summarize some basic facts about the Riemannian geometry of $SU_{2,m}/S(U_2 \cdot U_m)$. In Section 2 we obtain some basic geometric equations for real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$. In Section 3 we study real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ with Ricci commuting for $\xi \in \mathcal{Q}$, and in Section 4 those with Ricci commuting for $\xi \in \mathcal{Q}^\perp$. Finally in Section 5 we use these results to derive our classification.

1. THE COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIAN $SU_{2,m}/S(U_2 \cdot U_m)$

In this section we summarize basic material about complex hyperbolic Grassmann manifolds $SU_{2,m}/S(U_2 \cdot U_m)$, for details we refer to [1], [3], [18] and [19].

The Riemannian symmetric space $SU_{2,m}/S(U_2 \cdot U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space \mathbb{C}_2^{m+2} , becomes a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2 \cdot U_m)$, and denote by \mathfrak{g} and \mathfrak{k} the corresponding Lie algebra of the Lie group G and K respectively. Let B be the Killing form of \mathfrak{g} and denote by \mathfrak{p} the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to B . The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of \mathfrak{g} . The Cartan involution $\theta \in \text{Aut}(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m} A I_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 and I_m denotes the identity (2×2) -matrix and $(m \times m)$ -matrix respectively. Then $\langle X, Y \rangle = -B(X, \theta Y)$ becomes a positive definite $\text{Ad}(K)$ -invariant inner product on \mathfrak{g} . Its restriction to \mathfrak{p} induces a metric g on $SU_{2,m}/S(U_2 \cdot U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2 \cdot U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2 \cdot U_m)$ together with this particular Riemannian metric g .

The Lie algebra \mathfrak{k} decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where \mathfrak{u}_1 is the one-dimensional center of \mathfrak{k} . The adjoint action of \mathfrak{su}_2 on \mathfrak{p} induces the quaternionic Kähler structure \mathfrak{J} on $SU_{2,m}/S(U_2 \cdot U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2 \cdot U_m)$.

By construction, J commutes with each almost Hermitian structure J_ν in \mathfrak{J} for $\nu = 1, 2, 3$. Recall that a canonical local basis J_1, J_2, J_3 of a quaternionic Kähler structure \mathfrak{J} consists of three almost Hermitian structures J_1, J_2, J_3 in \mathfrak{J} such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν , which is locally defined on $SU_{2,m}/S(U_2 \cdot U_m)$, is selfadjoint and satisfies $(JJ_\nu)^2 = I$ and $\text{tr}(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_o SU_{2,m}/S(U_2 \cdot U_m)$ of $SU_{2,m}/S(U_2 \cdot U_m)$ at o with \mathfrak{p} in the usual way. Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . Since $SU_{2,m}/S(U_2 \cdot U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_o SU_{2,m}/S(U_2 \cdot U_m) \cong \mathfrak{p}$ is contained in some maximal

abelian subspace of \mathfrak{p} . Generically this subspace is uniquely determined by X , in which case X is called regular.

If there exists more than one maximal abelian subspaces of \mathfrak{p} containing X , then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Up to scaling there exists a unique $S(U(2) \cdot U(m))$ -invariant Riemannian metric g on $SU_{2,m}/S(U_2 \cdot U_m)$. Equipped with this metric $SU_{2,m}/S(U_2 \cdot U_m)$ is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler.

For computational reasons we normalize g such that the minimal sectional curvature of $(SU_{2,m}/S(U_2 \cdot U_m), g)$ is -4 . The sectional curvature K of the noncompact symmetric space $SU_{2,m}/S(U_2 \cdot U_m)$ equipped with the Killing metric g is bounded by $-4 \leq K \leq 0$. The sectional curvature -4 is obtained for all 2-planes $\mathbb{C}X$ when X is a non-zero vector with $JX \in \mathfrak{J}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1 \cdot U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4 .

When $m = 2$, we note that the isomorphism $SO(4, 2) \simeq SU(2, 2)$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2 \cdot U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space \mathbb{R}_2^6 . For this reason we assume $m \geq 3$ from now on, although many of the subsequent results also hold for $m = 1, 2$.

The Riemannian curvature tensor \bar{R} of $SU_{2,m}/S(U_2 \cdot U_m)$ is locally given by

$$\begin{aligned}
 \bar{R}(X, Y)Z = & -\frac{1}{2} \left[g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \right. \\
 & - g(JX, Z)JY - 2g(JX, Y)JZ \\
 & + \sum_{\nu=1}^3 \{ g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y \\
 & - 2g(J_\nu X, Y)J_\nu Z \} \\
 & \left. + \sum_{\nu=1}^3 \{ g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY \} \right],
 \end{aligned}
 \tag{1.1}$$

where J_1, J_2, J_3 is any canonical local basis of \mathfrak{J} .

Recall that a maximal flat in a Riemannian symmetric space \bar{M} is a connected complete flat totally geodesic submanifold of maximal dimension. A non-zero tangent vector X of \bar{M} is singular if X is tangent to more than one maximal flat in \bar{M} , otherwise X is regular. The singular tangent vectors of $SU_{2,m}/S(U_2 \cdot U_m)$ are precisely the eigenvectors and the asymptotic vectors of the self-adjoint endomorphisms JJ_1 , where J_1 is any almost Hermitian structure in \mathfrak{J} . In other words, a tangent vector X to $SU_{2,m}/S(U_2 \cdot U_m)$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Now we want to focus on a singular vector X of type $JX \in \mathfrak{J}X$. In this paper, we will have to compute explicitly Jacobi vector fields along geodesics whose tangent vectors are all singular of type $JX \in \mathfrak{J}X$. For this we need the eigenvalues and

eigenspaces of the Jacobi operator $\bar{R}_X := \bar{R}(\cdot, X)X$. Let X be a singular unit vector tangent to $SU_{2,m}/S(U_2 \cdot U_m)$ of type $JX \in \mathfrak{J}X$. Then there exists an almost Hermitian structure J_1 in \mathfrak{J} such that $JX = J_1X$ and the eigenvalues, eigenspaces and multiplicities of \bar{R}_X are respectively given by

principal curvature	eigenspace	multiplicity
0	$\mathbb{R}X \oplus \{Y Y \perp \mathbb{H}X, JY = -J_1Y\}$	$2m - 1$
-1	$\mathbb{H}X \ominus \mathbb{C}X \oplus \{Y Y \perp \mathbb{H}X, JY = J_1Y\}$	$2m$
-4	$\mathbb{R}JX$	1

where $\mathbb{R}X$, $\mathbb{C}X$ and $\mathbb{H}X$ denotes the real, complex and quaternionic span of X , respectively, and $\mathbb{C}^\perp X$ the orthogonal complement of $\mathbb{C}X$ in $\mathbb{H}X$.

The maximal totally geodesic submanifolds in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ are $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$, $\mathbb{C}H^m$, $\mathbb{C}H^k \times \mathbb{C}H^{m-k}$ ($1 \leq k \leq [m/2]$), $G_2^*(\mathbb{R}^{m+2})$ and $\mathbb{H}H^n$ (if $m = 2n$). The first three are complex submanifolds and the other two are real submanifolds with respect to the Kähler structure J . The tangent spaces of the totally geodesic $\mathbb{C}H^m$ are precisely the maximal linear subspaces of the form $\{X|JX = J_1X\}$ with some fixed almost Hermitian structure $J_1 \in \mathfrak{J}$.

2. REAL HYPERSURFACES IN $SU_{2,m}/S(U_2 \cdot U_m)$

Let M be a real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, that is, a hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with real codimension one. The induced Riemannian metric on M will also be denoted by g , and ∇ denotes the Levi Civita covariant derivative of (M, g) . We denote by \mathcal{C} and \mathcal{Q} the maximal complex and quaternionic subbundle of the tangent bundle TM of M , respectively. Now let us put

$$(2.1) \quad JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector field X of a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$, where ϕX denotes the tangential component of JX and N a unit normal vector field of M in $SU_{2,m}/S(U_2 \cdot U_m)$.

From the Kähler structure J of $SU_{2,m}/S(U_2 \cdot U_m)$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$(2.2) \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \text{and} \quad \eta(X) = g(X, \xi)$$

for any vector field X on M and $\xi = -JN$.

If M is orientable, then the vector field ξ is globally defined and said to be the induced *Reeb vector field* on M . Furthermore, let J_1, J_2, J_3 be a canonical local basis of \mathfrak{J} . Then each J_ν induces a local almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, on M . Locally, \mathcal{C} is the orthogonal complement in TM of the real span of ξ , and \mathcal{Q} the orthogonal complement in TM of the real span of $\{\xi_1, \xi_2, \xi_3\}$.

Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of \mathfrak{J} . Then the quaternionic Kähler structure J_ν of $SU_{2,m}/S(U_2 \cdot U_m)$, together with the condition

$$J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$$

in section 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned} (2.3) \quad & \phi_\nu^2 X = -X + \eta_\nu(X)(\xi_\nu), \quad \phi_\nu \xi_\nu = 0, \quad \eta_\nu(\xi_\nu) = 1 \\ & \phi_{\nu+1} \xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\ & \phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X) \xi_\nu, \\ & \phi_{\nu+1} \phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X) \xi_{\nu+1} \end{aligned}$$

for any vector field X tangent to M . The tangential and normal component of the commuting identity $JJ_\nu X = J_\nu JX$ give

$$(2.4) \quad \phi \phi_\nu X - \phi_\nu \phi X = \eta_\nu(X) \xi - \eta(X) \xi_\nu \quad \text{and} \quad \eta_\nu(\phi X) = \eta(\phi_\nu X).$$

The last equation implies $\phi_\nu \xi = \phi \xi_\nu$. The tangential and normal component of $J_\nu J_{\nu+1} X = J_{\nu+2} X = -J_{\nu+1} J_\nu X$ give

$$(2.5) \quad \phi_\nu \phi_{\nu+1} X - \eta_{\nu+1}(X) \xi_\nu = \phi_{\nu+2} X = -\phi_{\nu+1} \phi_\nu X + \eta_\nu(X) \xi_{\nu+1}$$

and

$$(2.6) \quad \eta_\nu(\phi_{\nu+1} X) = \eta_{\nu+2}(X) = -\eta_{\nu+1}(\phi_\nu X).$$

Putting $X = \xi_\nu$ and $X = \xi_{\nu+1}$ into the first of these two equations yields $\phi_{\nu+2} \xi_\nu = \xi_{\nu+1}$ and $\phi_{\nu+2} \xi_{\nu+1} = -\xi_\nu$ respectively. Using the Gauss and Weingarten formulas, the tangential and normal component of the Kähler condition $(\bar{\nabla}_X J)Y = 0$ give $(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$ and $(\nabla_X \eta)Y = g(\phi AX, Y)$. The last equation implies $\nabla_X \xi = \phi AX$. Finally, using the explicit expression for the Riemannian curvature tensor \bar{R} of $SU_{2,m}/S(U_2 \cdot U_m)$ in [3] the Codazzi equation takes the form

$$\begin{aligned} (2.7) \quad & (\nabla_X A)Y - (\nabla_Y A)X = -\frac{1}{2} \left[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \right. \\ & + \sum_{\nu=1}^3 \{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \} \\ & + \sum_{\nu=1}^3 \{ \eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X \} \\ & \left. + \sum_{\nu=1}^3 \{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \} \xi_\nu \right]. \end{aligned}$$

for any vector fields X and Y on M . Moreover, by the expression of the curvature tensor (1.1), we have the equation of Gauss as follows:

$$\begin{aligned}
 (2.8) \quad R(X, Y)Z = & -\frac{1}{2} \left[g(Y, Z)X - g(X, Z)Y \right. \\
 & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\
 & + \sum_{\nu=1}^3 \{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\
 & + \sum_{\nu=1}^3 \{g(\phi_\nu \phi Y, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y\} \\
 & - \sum_{\nu=1}^3 \{\eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y\} \\
 & \left. - \sum_{\nu=1}^3 \{\eta(X)g(\phi_\nu \phi Y, Z) - \eta(Y)g(\phi_\nu \phi X, Z)\}\xi_\nu \right] \\
 & + g(AY, Z)AX - g(AX, Z)AY
 \end{aligned}$$

for any vector fields X, Y, Z and W on M . Hereafter, unless otherwise stated, we want to use these basic equations mentioned above frequently without referring to them explicitly.

3. SOME PRELIMINARIES IN $SU_{2,m}/S(U_2 \cdot U_m)$

In this section we can introduce some preliminaries in $SU_{2,m}/S(U_2 \cdot U_m)$ corresponding to the formulas given in [18] from the negative curvature tensor (2.8). Now let us contract Y and Z in the equation of Gauss (2.8) in section 2. Then the curvature tensor for a real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ gives a Ricci tensor defined by

$$\begin{aligned}
 (3.1) \quad SX = & \sum_{i=1}^{4m-1} R(X, e_i)e_i \\
 = & -\frac{1}{2} \left[(4m+10)X - 3\eta(X)\xi - 3 \sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu \right. \\
 & + \sum_{\nu=1}^3 \{(\text{Tr} \phi_\nu \phi)\phi_\nu \phi X - (\phi_\nu \phi)^2 X\} \\
 & - \sum_{\nu=1}^3 \{\eta_\nu(\xi)\phi_\nu \phi X - \eta(X)\phi_\nu \phi \xi_\nu\} \\
 & \left. - \sum_{\nu=1}^3 \{(\text{Tr} \phi_\nu \phi)\eta(X) - \eta(\phi_\nu \phi X)\}\xi_\nu \right] + hAX - A^2X,
 \end{aligned}$$

where h denotes the trace of the shape operator A of M in $SU_{2,m}/S(U_2 \cdot U_m)$. From the formula $JJ_\nu = J_\nu J$, $\text{Tr} JJ_\nu = 0$, $\nu = 1, 2, 3$ we calculate the following for any

basis $\{e_1, \dots, e_{4m-1}, N\}$ of the tangent space of $SU_{2,m}/S(U_2 \cdot U_m)$

$$\begin{aligned}
 0 &= \text{Tr } JJ_\nu \\
 &= \sum_{k=1}^{4m-1} g(JJ_\nu e_k, e_k) + g(JJ_\nu N, N) \\
 &= \text{Tr } \phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN) \\
 &= \text{Tr } \phi\phi_\nu - 2\eta_\nu(\xi)
 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
 (\phi_\nu \phi)^2 X &= \phi_\nu \phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu) \\
 &= \phi_\nu(-\phi_\nu X + \eta(\phi_\nu X)\xi) + \eta(X)\phi_\nu^2 \xi \\
 &= X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu \xi \\
 &\quad + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}.
 \end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$\begin{aligned}
 SX &= -\frac{1}{2} \left[(4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu \right. \\
 &\quad \left. + \sum_{\nu=1}^3 \{ \eta_\nu(\xi)\phi_\nu \phi X - \eta(\phi_\nu X)\phi_\nu \xi - \eta(X)\eta_\nu(\xi)\xi_\nu \} \right] \\
 &\quad + hAX - A^2 X.
 \end{aligned} \tag{3.4}$$

Now the covariant derivative of (3.4) becomes

$$\begin{aligned}
 (\nabla_Y S)X &= \frac{3}{2}((\nabla_Y \eta)X)\xi + \frac{3}{2}\eta(X)\nabla_Y \xi \\
 &\quad + \frac{3}{2}\sum_{\nu=1}^3 (\nabla_Y \eta_\nu)(X)\xi_\nu + \frac{3}{2}\sum_{\nu=1}^3 \eta_\nu(X)\nabla_Y \xi_\nu \\
 &\quad - \frac{1}{2}\sum_{\nu=1}^3 \left\{ Y(\eta_\nu(\xi))\phi_\nu \phi X + \eta_\nu(\xi)(\nabla_Y \phi_\nu)\phi X \right. \\
 &\quad + \eta_\nu(\xi)\phi_\nu(\nabla_Y \phi)X - (\nabla_Y \eta)(\phi_\nu X)\phi_\nu \xi \\
 &\quad - \eta((\nabla_Y \phi_\nu)X)\phi_\nu \xi - \eta(\phi_\nu X)\nabla_Y(\phi_\nu \xi) \\
 &\quad - (\nabla_Y \eta)(X)\eta_\nu(\xi)\xi_\nu - \eta(X)\nabla_Y(\eta_\nu(\xi))\xi_\nu \\
 &\quad \left. - \eta(X)\eta_\nu(\xi)\nabla_Y \xi_\nu \right\} \\
 &\quad + (Yh)AX + h(\nabla_Y A)X - (\nabla_Y A^2)X
 \end{aligned}$$

for any vector fields X and Y tangent to M in $SU_{2,m}/S(U_2 \cdot U_m)$. Then from the above formula, together with the formulas in section 2, we have

$$\begin{aligned}
(\nabla_Y S)X &= \frac{3}{2}g(\phi AY, X)\xi + \frac{3}{2}\eta(X)\phi AY \\
&+ \frac{3}{2}\sum_{\nu=1}^3\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) \\
&+ g(\phi_\nu AY, X)\}\xi_\nu \\
&+ \frac{3}{2}\sum_{\nu=1}^3\eta_\nu(X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu AY\} \\
&- \frac{1}{2}\sum_{\nu=1}^3\left[Y(\eta_\nu(\xi))\phi_\nu\phi X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi X \right. \\
(3.5) \quad &+ q_{\nu+2}(Y)\phi_{\nu+1}\phi X + \eta_\nu(\phi X)AY - g(AY, \phi X)\xi_\nu\} \\
&+ \eta_\nu(\xi)\{\eta(X)\phi_\nu AY - g(AY, X)\phi_\nu\xi\} - g(\phi AY, \phi_\nu X)\phi_\nu\xi \\
&+ \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X) - \eta_\nu(X)\eta(AY) \\
&+ \eta(\xi_\nu)g(AY, X)\}\phi_\nu\xi \\
&- \eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi + \phi_\nu\phi AY \\
&- \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\} \\
&- g(\phi AY, X)\eta_\nu(\xi)\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\xi_\nu - \eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\Big] \\
&+ (Yh)AX + h(\nabla_Y A)X - (\nabla_Y A^2)X.
\end{aligned}$$

Now let us take a covariant derivative of $S\phi = \phi S$. Then it gives that

$$(3.6) \quad (\nabla_Y S)\phi X + S(\nabla_Y \phi)X = (\nabla_Y \phi)SX + \phi(\nabla_Y S)X.$$

Then the first term of (3.6) becomes

$$\begin{aligned}
 (\nabla_Y S)\phi X &= \frac{3}{2}g(\phi AY, \phi X)\xi \\
 &+ \frac{3}{2}\sum_{\nu=1}^3\{q_{\nu+2}(Y)\eta_{\nu+1}(\phi X) - q_{\nu+1}(Y)\eta_{\nu+2}(\phi X) + g(\phi_\nu AY, \phi X)\}\xi_\nu \\
 &+ \frac{3}{2}\sum_{\nu=1}^3\eta_\nu(\phi X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu A\phi X\} \\
 &- \frac{1}{2}\sum_{\nu=1}^3\left[Y(\eta_\nu(\xi))\phi_\nu\phi^2 X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi^2 X \right. \\
 (3.7) \quad &+ q_{\nu+2}(Y)\phi_{\nu+1}\phi^2 X + \eta_\nu(\phi^2 X)AY - g(AY, \phi^2 X)\xi_\nu\} \\
 &- \eta_\nu(\xi)g(AY, \phi X)\phi_\nu\xi - g(\phi AY, \phi_\nu\phi X)\phi_\nu\xi \\
 &+ \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}\phi X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}\phi X) - \eta_\nu(\phi X)\eta(AY) \\
 &+ \eta(\xi_\nu)g(AY, \phi X)\}\phi_\nu\xi \\
 &- \eta(\phi_\nu\phi X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi \\
 &+ \phi_\nu\phi AY - \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\} - g(\phi AY, \phi X)\eta_\nu(\xi)\xi_\nu\Big] \\
 &+ (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X.
 \end{aligned}$$

The second term of (3.6) becomes

$$\begin{aligned}
 S(\nabla_Y \phi)X &= S\{\eta(X)AY - g(AY, X)\xi\} \\
 &= \eta(X)\left[-\frac{1}{2}\left\{(4m+7)AY - 3\eta(AY)\xi - 3\sum_{\nu=1}^3\eta_\nu(AY)\xi_\nu \right. \right. \\
 (3.8) \quad &+ \sum_{\nu=1}^3\{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu AY)\phi_\nu\xi - \eta(AY)\eta_\nu(\xi)\xi_\nu\}\Big] \\
 &+ hA^2Y - A^3Y\Big] \\
 &- g(AY, X)\left[-\frac{1}{2}\left\{(4m+7)\xi - 3\xi - 4\sum_{\nu=1}^3\eta_\nu(\xi)\xi_\nu\right\} + hA\xi - A^2\xi\right].
 \end{aligned}$$

The first term of the right side in (3.6) becomes

$$(\nabla_Y \phi)SX = \eta(SX)AY - g(AY, SX)\xi,$$

and the second term of the right side in (3.6) is given by

(3.9)

$$\begin{aligned}
\phi(\nabla_Y S)X &= \frac{3}{2}\eta(X)\phi^2 AY \\
&+ \frac{3}{2}\sum_{\nu=1}^3 \{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, \phi X)\}\phi\xi_\nu \\
&+ \frac{3}{2}\sum_{\nu=1}^3 \eta_\nu(X)\{q_{\nu+2}(Y)\phi\xi_{\nu+1} - q_{\nu+1}(Y)\phi\xi_{\nu+2} + \phi\phi_\nu AY\} \\
&- \frac{1}{2}\sum_{\nu=1}^3 \left[Y(\eta_\nu(\xi))\phi\phi_\nu\phi X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi\phi_{\nu+2}\phi X + q_{\nu+2}(Y)\phi\phi_{\nu+1}\phi X \right. \\
&\quad + \eta_\nu(\phi X)\phi AY - g(AY, \phi X)\phi\xi_\nu\} \\
&\quad + \eta_\nu(\xi)\{\eta(X)\phi\phi_\nu AY - g(AY, X)\phi\phi_\nu\xi\} - g(\phi AY, \phi_\nu X)\phi\phi_\nu\xi \\
&\quad + \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X) \\
&\quad - \eta_\nu(X)\eta(AY) + \eta(\xi_\nu)g(AY, X)\}\phi\phi_\nu\xi \\
&\quad - \eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi\phi_{\nu+2}\xi + \phi\phi_\nu\phi AY \\
&\quad - \eta(AY)\phi\xi_\nu + \eta(\xi_\nu)\phi AY\} \\
&\quad - g(\phi AY, X)\eta_\nu(\xi)\phi\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\phi\xi_\nu - \eta(X)\eta_\nu(\xi)\phi\nabla_Y\xi_\nu \Big] \\
&+ (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X.
\end{aligned}$$

Putting $X = \xi$ into (3.5) and using that the Reeb vector field ξ is principal, that is, $A\xi = \alpha\xi$, then we have

(3.10)

$$\begin{aligned}
S(\nabla_Y \phi)\xi &= \left[-\frac{1}{2}\left\{ (4m+7)AY - 3\eta(AY)\xi - 3\sum_{\nu=1}^3 \eta_\nu(AY)\xi_\nu \right. \right. \\
&\quad \left. \left. + \sum_{\nu=1}^3 \{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu\phi AY)\phi_\nu\xi - \alpha\eta(Y)\eta_\nu(\xi)\xi_\nu\} \right\} + hA^2Y - A^3Y \right] \\
&\quad - \alpha\eta(Y)\left[-\frac{1}{2}\left\{ 4(m+1)\xi - 4\sum_{\nu=1}^3 \eta_\nu(\xi)\xi_\nu \right\} + (\alpha h - \alpha^2)\xi \right].
\end{aligned}$$

Moreover, the right side of (3.6) becomes

$$\begin{aligned}
 & (\nabla_Y \phi)S\xi + \phi(\nabla_Y S)\xi \\
 &= \eta(S\xi)AY - g(AY, S\xi)\xi + \phi(\nabla_Y S)\xi \\
 &= \left[\{-2(m+1) + h\alpha - \alpha^2\} + 2\sum_{\nu=1}^3 \eta_\nu(\xi)^2 \right] AY + \frac{3}{2}\phi^2 AY \\
 &\quad - \left[\{-2(m+1)\alpha + h\alpha^2 - \alpha^3\}\eta(Y) + 2\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(AY) \right] \xi \\
 (3.11) \quad &+ \frac{3}{2}\sum_{\nu=1}^3 \{q_{\nu+2}(Y)\eta_{\nu+1}(\xi) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi) + \eta_\nu(\phi AY)\}\phi\xi_\nu \\
 &+ \frac{3}{2}\sum_{\nu=1}^3 \eta_\nu(\xi)\{q_{\nu+2}(Y)\phi\xi_{\nu+1} - q_{\nu+1}(Y)\phi\xi_{\nu+2} + \phi\phi_\nu AY\} \\
 &- \frac{1}{2}\sum_{\nu=1}^3 \left[\eta_\nu(\xi)\{\phi\phi_\nu AY - \alpha\eta(Y)\phi^2\xi_\nu\} - g(\phi AY, \phi\xi_\nu)\phi^2\xi_\nu \right. \\
 &\quad \left. - Y(\eta_\nu(\xi))\phi\xi_\nu - \eta_\nu(\xi)\phi\nabla_Y \xi_\nu \right] + h\phi(\nabla_Y A)\xi - \phi(\nabla_Y A^2)\xi.
 \end{aligned}$$

From this, putting $Y = \xi$ into (3.11), we obtain

$$\begin{aligned}
 0 &= \sum_{\nu=1}^3 \{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi\xi_\nu \\
 (3.12) \quad &+ \sum_{\nu=1}^3 \eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi\xi_{\nu+1} - q_{\nu+1}(\xi)\phi\xi_{\nu+2} + \alpha\phi^2\xi_\nu\}.
 \end{aligned}$$

Now in order to show that ξ belongs to either the distribution \mathcal{Q} or to the distribution \mathcal{Q}^\perp , let us assume that $\xi = X_1 + X_2$ for some $X_1 \in \mathcal{Q}$ and $X_2 \in \mathcal{Q}^\perp$. Then it follows that

$$\begin{aligned}
 0 &= \sum_{\nu=1}^3 \left\{ q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi) \right\} (\phi_\nu X_1 + \phi_\nu X_2) \\
 (3.13) \quad &+ \sum_{\nu=1}^3 \eta_\nu(\xi) \left\{ q_{\nu+2}(\xi)(\phi_{\nu+1} X_1 + \phi_{\nu+1} X_2) \right. \\
 &\quad \left. - q_{\nu+1}(\xi)(\phi_{\nu+2} X_1 + \phi_{\nu+2} X_2) - \alpha\xi_\nu + \alpha\eta(\xi_\nu)(X_1 + X_2) \right\}.
 \end{aligned}$$

Then by comparing \mathcal{Q} and \mathcal{Q}^\perp component of (3.13), we have respectively

$$\begin{aligned}
 0 &= \sum_{\nu=1}^3 \{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi_\nu X_1 + \alpha \sum_{\nu=1}^3 \eta_\nu(\xi)^2 X_1 \\
 (3.14) \quad &+ \sum_{\nu=1}^3 \eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi_{\nu+1} X_1 - q_{\nu+1}(\xi)\phi_{\nu+2} X_1\},
 \end{aligned}$$

$$\begin{aligned}
(3.15) \quad 0 &= \sum_{\nu=1}^3 \{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi_{\nu}X_2 \\
&\quad + \sum_{\nu=1}^3 \eta_{\nu}(\xi)\{q_{\nu+2}(\xi)\phi_{\nu+1}X_2 - q_{\nu+1}(\xi)\phi_{\nu+2}X_2 - \alpha\xi_{\nu} + \alpha\eta(\xi_{\nu})X_2\}.
\end{aligned}$$

Taking an inner product (3.14) with X_1 , we have

$$(3.16) \quad \alpha \sum_{\nu=1}^3 \eta_{\nu}(\xi)^2 = 0.$$

Then $\alpha = 0$ or $\eta_{\nu}(\xi) = 0$ for $\nu = 1, 2, 3$. So for a non-vanishing geodesic Reeb flow we have $\eta_{\nu}(\xi) = 0$, $\nu = 1, 2, 3$. This means that $\xi \in \mathcal{Q}$, which contradicts to our assumption $\xi = X_1 + X_2$. Including this one, we are able to assert the following:

Lemma 3.1. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor. Then the Reeb vector ξ belongs to either the distribution \mathcal{Q} or to the distribution \mathcal{Q}^{\perp} .*

Proof. When the geodesic Reeb flow is non-vanishing, that is $\alpha \neq 0$, (3.16) gives $\xi \in \mathcal{Q}$. When the geodesic Reeb flow is vanishing, we differentiate $A\xi = 0$. Then by Suh ([17] and [18]) we know that

$$\sum_{\nu=1}^3 \eta_{\nu}(\xi)\eta_{\nu}(\phi Y) = 0.$$

From this, by replacing $Y \in \mathcal{Q}$ by ϕY , it follows that

$$\sum_{\nu=1}^3 \eta_{\nu}^2(\xi)\eta(Y) = 0.$$

So if there are some $Y \in \mathcal{Q}$ such that $\eta(Y) \neq 0$, then $\eta_{\nu}(\xi) = 0$ for $\nu = 1, 2, 3$. This means that $\xi \in \mathcal{Q}$. If $\eta(Y) = 0$ for any $Y \in \mathcal{Q}$, then we know $\xi \in \mathcal{Q}^{\perp}$. \square

4. REAL HYPERSURFACES WITH GEODESIC REEB FLOW SATISFYING $\xi \in \mathcal{Q}$

Now in this section let us show that the distribution \mathcal{Q} of a Hopf real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ with $\xi \in \mathcal{Q}$ satisfies $g(A\mathcal{Q}, \mathcal{Q}^{\perp}) = 0$.

The Reeb vector ξ is said to be a *Hopf* vector if it is a principal vector for the shape operator A of M in $SU_{2,m}/S(U_2 \cdot U_m)$, that is, the Reeb vector ξ is invariant under the shape operator A .

In a theorem due to Berndt and Suh [3] we know that the Reeb vector ξ of M belongs to the maximal quaternionic subbundle \mathcal{Q} when M is locally congruent to a real hypersurface of type (B), that is, a tube over a totally real totally geodesic $\mathbb{H}P^n$, $m = 2n$, or a horosphere in $SU_{2,m}/S(U_2 U_m)$ with $JN \perp \mathfrak{J}N$ whose center at infinity is singular, and a real hypersurface of type (D) in $SU_{2,m}/S(U_2 \cdot U_m)$. Naturally we are able to consider a converse problem. From such a view point Suh [20] has proved the following for real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ with the Reeb vector field $\xi \in \mathcal{Q}$.

Theorem C. *Let M be a real hypersurface in noncompact complex two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ with the Reeb vector field belonging to the maximal quaternionic subbundle \mathcal{Q} . Then one of the following statements holds,*

- (B) *M is an open part of a tube around a totally geodesic $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2 U_{2n})$, $m = 2n$,*
- (C₂) *M is an open part of a horosphere in $SU_{2,m}/S(U_2 U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$,*
or the following exceptional case holds:
- (D) *The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$. Moreover, M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \sqrt{2}, \gamma = 0, \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), T_\gamma = J(TM \ominus \mathcal{Q}), T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then we have $T_\mu \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}$, $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

In the proof of Theorem C we have used the equation of Codazzi in section 2 for Hopf real hypersurfaces M in $SU_{2,m}/S(U_2 \cdot U_m)$, and proved that the quaternionic maximal subbundle \mathcal{Q} is invariant under the shape operator, that is, $g(A\mathcal{Q}, \mathcal{Q}^\perp) = 0$, if the Reeb vector field ξ belongs to the subbundle \mathcal{Q} of M . So by using a theorem due to Berndt and Suh [3] we can assert Theorem B in the introduction. Then among the classification of Theorem A the Reeb vector field ξ of real hypersurfaces in Theorem C belongs to the maximal quaternionic subbundle \mathcal{Q} .

Now let us check whether the Ricci tensor S of hypersurfaces mentioned in Theorem C satisfies the commuting condition or not. In order to do this, we should find all of the principal curvatures corresponding to the hypersurfaces in Theorem B. For cases of type (B), one of type (C) which will be said to be of type (C₂), and of type (D) in Theorem B let us introduce a proposition due to Berndt and Suh [5] as follows:

Proposition 4.1. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2 U_m)$, $m \geq 2$. Assume that the maximal complex subbundle \mathcal{C} of TM and the maximal quaternionic subbundle \mathcal{Q} of TM are both invariant under the shape operator of M . If $JN \perp \mathfrak{J}N$, then one of the following statements holds:*

- (B) *M has five (four for $r = \sqrt{2} \operatorname{Arctanh}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures*

$$\begin{aligned} \alpha &= \sqrt{2} \tanh(\sqrt{2}r), \beta = \sqrt{2} \coth(\sqrt{2}r), \gamma = 0, \\ \lambda_1 &= \frac{1}{\sqrt{2}} \tanh\left(\frac{1}{\sqrt{2}}r\right), \lambda_2 = \frac{1}{\sqrt{2}} \coth\left(\frac{1}{\sqrt{2}}r\right), \end{aligned}$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C}, T_\beta = TM \ominus \mathcal{Q}, T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are invariant under \mathfrak{J} and are mapped onto each other by J . In particular, the quaternionic dimension of $SU_{2,m}/S(U_2 U_m)$ must be even.

(C₂) M has exactly three distinct constant principal curvatures

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), \quad T_\gamma = J(TM \ominus \mathcal{Q}), \quad T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

(D) M has at least four distinct principal curvatures, three of which are given by

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}), \quad T_\gamma = J(TM \ominus \mathcal{Q}), \quad T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$.

On the other hand, we calculate the following

$$\begin{aligned} S\phi X = & -\frac{1}{2} \left[(4m+7)\phi X - 3 \sum_{\nu=1}^3 \eta_\nu(\phi X) \xi_\nu + \sum_{\nu=1}^3 \{ \eta_\nu(\xi) \phi_\nu \phi^2 X - \eta(\phi_\nu \phi X) \phi_\nu \xi \right. \\ & \left. - \eta(\phi X) \eta_\nu(\xi) \xi_\nu \} \right] + hA\phi X - A^2\phi X \end{aligned}$$

and

$$\begin{aligned} \phi SX = & -\frac{1}{2} \left[(4m+7)\phi X - 3 \sum_{\nu=1}^3 \eta_\nu(X) \phi \xi_\nu + \sum_{\nu=1}^3 \{ \eta_\nu(\xi) \phi \phi_\nu \phi X - \eta(\phi_\nu X) \phi \phi_\nu \xi \right. \\ & \left. - \eta(X) \eta_\nu(\xi) \phi \xi_\nu \} \right] + h\phi AX - \phi A^2 X. \end{aligned}$$

Then the Ricci commuting $S\phi = \phi S$ implies that

$$(4.1) \quad hA\phi X - A^2\phi X = h\phi AX - \phi A^2 X + 2 \sum_{\nu=1}^3 \eta_\nu(X) \phi \xi_\nu - 2 \sum_{\nu=1}^3 \eta_\nu(\phi X) \xi_\nu.$$

First let us check that real hypersurfaces of type (B) in Theorem B satisfy the formula (4.1) or not.

Putting $X = \xi \in \mathcal{Q}$ we know the formula (4.1) holds. For $X = \xi_i \in \mathcal{Q}^\perp$ the left side of (4.1) becomes

$$L = hA\phi \xi_i - A^2\phi \xi_i = 0$$

and the right side is given by

$$\begin{aligned} R = & h\phi A \xi_i - \phi A^2 \xi_i + 2\eta_\nu(\xi_i) \phi \xi_\nu - 2 \sum_{\nu=1}^3 \eta_\nu(\phi \xi_i) \xi_\nu \\ = & h\beta \phi \xi_i - \beta^2 \phi \xi_i + 2\phi \xi_i = (h\beta - \beta^2 + 2)\phi \xi_i. \end{aligned}$$

Then $\beta(h - \beta) = -2$.

On the other hand, the trace h is given by

$$\begin{aligned} h = & \alpha + 3\beta + 3\gamma + 2(m-2)(\lambda_1 + \lambda_2) \\ = & \sqrt{2} \tanh(\sqrt{2}r) + (2m-1)\sqrt{2} \coth(\sqrt{2}r), \end{aligned}$$

where we have used $2 \coth \sqrt{2}r = \tanh(\frac{1}{\sqrt{2}}r) + \coth(\frac{1}{\sqrt{2}}r)$. Then it follows that

$$\begin{aligned}\beta(h - \beta) &= (\sqrt{2} \tanh(\sqrt{2}r) + (2m - 2)\sqrt{2} \coth(\sqrt{2}r))\sqrt{2} \coth(\sqrt{2}r) \\ &= 2 + 4(m - 1) \coth^2(\sqrt{2}r) \\ &= -2,\end{aligned}$$

which gives $4(m - 1) \coth^2(\sqrt{2}r) = -4$. This gives a contradiction.

As a second, let us check whether or not a horosphere at infinity could satisfy the Ricci commuting. By Proposition 4.1, we know that the principal curvatures of the case (C_2) are given by

$$\alpha = \beta = \sqrt{2}, \gamma = 0, \lambda = \frac{1}{\sqrt{2}}.$$

Then the trace of the shape operator h becomes

$$\begin{aligned}h &= \alpha + 3\beta + 3\gamma + 4(m - 2)\lambda \\ &= 4\sqrt{2} + 2(m - 2)\sqrt{2}\end{aligned}$$

Then by putting $X = \xi_i \in \mathcal{Q}^\perp$ in (4.1), we know that $(h\beta - \beta^2 + 2)\phi\xi_i = 0$. This gives that $-2 = \beta(h - \beta) = 2(2m - 1)$, which gives a contradiction.

As a third, we want to check that real hypersurfaces of type (D) in Theorem C satisfy the Ricci commuting or not. Then the principal curvature of type (D) becomes

$$\alpha = \beta = \sqrt{2}, \gamma = 0, \lambda = \frac{1}{\sqrt{2}} \quad \text{and} \quad \text{quad}\mu$$

where $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$. Then the trace h becomes that

$$\begin{aligned}h &= \alpha + 3\beta + 3\gamma + (2m - 4)\frac{1}{\sqrt{2}} + (2m - 4)\mu \\ &= (m - 2)(\sqrt{2} + 2\mu) + 4\sqrt{2}.\end{aligned}$$

From this, together with $\beta(h - \beta) = -2$, we have

$$\sqrt{2}\{(m - 2)(\sqrt{2} + 2\mu) + 3\sqrt{2}\} = -2.$$

This gives $2\mu(m - 2) = -(m + 2)\sqrt{2}$. Then the trace h becomes that

$$\begin{aligned}h &= (m - 2)(\sqrt{2} + 2\mu) + 4\sqrt{2} \\ &= (m - 2)\sqrt{2} - (m + 2)\sqrt{2} + 4\sqrt{2} \\ &= 0.\end{aligned}$$

On the other hand, from the Ricci commuting (4.1) we know that

$$hA\phi X - A^2\phi X = h\phi AX - \phi A^2X$$

for any $X \in \mathcal{Q}$. So it follows that $h = \lambda + \mu = -\frac{2\sqrt{2}}{m-2}$. But this is a contradiction. So also the case of type (D) can not occur.

Summing up all cases (B) , (C_2) and (D) mentioned above, we conclude that there do not exist any Hopf real hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor when the Reeb vector field ξ belongs to the distribution \mathcal{Q} .

5. REAL HYPERSURFACES WITH GEODESIC REEB FLOW SATISFYING $\xi \in \mathcal{Q}^\perp$

Now let us consider a Hopf real hypersurface M in $SU_{2,m}/S(U_2 \cdot U_m)$ with commuting Ricci tensor satisfying $\xi \in \mathcal{Q}^\perp$, where \mathcal{Q} denotes a quaternionic maximal subbundle in $T_x M$, $x \in M$ such that $T_x M = \mathcal{Q} \oplus \mathcal{Q}^\perp$. Since we assume $\xi \in \mathcal{Q}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$, there exists a Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1 N$, that is, $\xi = \xi_1$.

Moreover, the right side of (3.6) can be written as follows:

$$\begin{aligned}
 & (\nabla_Y \phi)S\xi + \phi(\nabla_Y S)\xi \\
 &= \eta(S\xi)AY - g(AY, S\xi)\xi + \phi(\nabla_Y S)\xi \\
 &= \left[\{-2(m+1) + h\alpha - \alpha^2\} + 2 \sum_{\nu=1}^3 \eta_\nu(\xi)^2 \right] AY + \frac{3}{2} \phi^2 AY \\
 &\quad - \left[\{-2(m+1)\alpha + h\alpha^2 - \alpha^3\} \eta(Y) + 2 \sum_{\nu=1}^3 \eta_\nu(\xi) \eta_\nu(AY) \right] \xi \\
 (5.1) \quad &+ \frac{3}{2} \sum_{\nu=1}^3 \{q_{\nu+2}(Y) \eta_{\nu+1}(\xi) - q_{\nu+1}(Y) \eta_{\nu+2}(\xi) + \eta_\nu(\phi AY)\} \phi \xi_\nu \\
 &+ \frac{3}{2} \sum_{\nu=1}^3 \eta_\nu(\xi) \{q_{\nu+2}(Y) \phi \xi_{\nu+1} - q_{\nu+1}(Y) \phi \xi_{\nu+2} + \phi \phi_\nu AY\} \\
 &- \frac{1}{2} \sum_{\nu=1}^3 \left[\eta_\nu(\xi) \{ \phi \phi_\nu AY - \alpha \eta(Y) \phi^2 \xi_\nu \} - g(\phi AY, \phi \xi_\nu) \phi^2 \xi_\nu \right. \\
 &\quad \left. - Y(\eta_\nu(\xi)) \phi \xi_\nu - \eta_\nu(\xi) \phi \nabla_Y \xi_\nu \right] + h \phi(\nabla_Y A)\xi - \phi(\nabla_Y A^2)\xi.
 \end{aligned}$$

Then $(\nabla_Y S)\phi\xi + S(\nabla_Y \phi)\xi = (\nabla_Y \phi)S\xi + \phi(\nabla_Y S)\xi$ and $\xi \in \mathcal{Q}^\perp$, that is, $\xi = \xi_1$, becomes

$$\begin{aligned}
 & -\frac{1}{2} \left\{ (4m+7)AY - 6\alpha\eta(Y)\xi - 3\eta_2(AY)\xi_2 - 3\eta_3(AY)\xi_3 \right. \\
 &\quad \left. + \phi_1 \phi AY - \eta(\phi_2 AY)\phi_2 \xi - \eta(\phi_3 AY)\phi_3 \xi - \alpha\eta(Y)\xi \right\} \\
 (5.2) \quad &+ hA^2Y - A^3Y - \alpha\eta(Y)\{-2m\xi + (\alpha h - \alpha^2)\xi\} \\
 &= \{-2m + h\alpha - \alpha^2\}AY - \{-2m + h\alpha - \alpha^2\}AY \\
 &\quad - \{-2m + h\alpha - \alpha^2\}\alpha\eta(Y)\xi \\
 &+ \frac{3}{2} \left[\phi^2 AY + (q_1(Y)\eta_3(\xi) - q_3(Y)\eta(\xi)) \right. \\
 &\quad + \eta_2(\phi AY)\phi \xi_2 + (q_2(Y)\eta(\xi) - q_1(Y)\eta_2(\xi) + \eta_3(\phi AY)\phi \xi_3)\phi \xi_3 \\
 &\quad \left. + (q_3(Y)\phi \xi_2 - q_2(Y)\phi \xi_3 + \phi \phi_1 AY) \right] \\
 &- \frac{1}{2} \left[\phi \phi_1 AY - \alpha\eta(Y)\phi \phi_1 \xi - g(\phi AY, \phi_2 \xi)\phi \phi_2 \xi - g(\phi AY, \phi_3 \xi)\phi \phi_3 \xi \right. \\
 &\quad \left. - \phi^2 AY \right] + h\alpha \phi^2 AY - h\phi A \phi AY - \alpha^2 \phi^2 AY + \phi A^2 \phi AY.
 \end{aligned}$$

Then (5.2) can be rearranged as follows:

$$(5.3) \quad \begin{aligned} & -5AY + 5\alpha\eta(Y)\xi + 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 + 2hA^2Y - 2A^3Y \\ & = 3\phi\phi_1AY + 2(h\alpha^2 - \alpha^3)\eta(Y)\xi - 2h\phi A\phi AY + 2\phi A^2\phi AY. \end{aligned}$$

Now let us use a Proposition due to Suh (see [19]) as follows:

Proposition 5.1. *If M is a connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ with geodesic Reeb flow, then*

$$\begin{aligned} & 2g(A\phi AX, Y) - \alpha g((A\phi + \phi A)X, Y) + g(\phi X, Y) \\ & = \sum_{\nu=1}^3 \left\{ \eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi) \right. \\ & \quad \left. - 2\eta(X)\eta_\nu(\phi Y)\eta_\nu(\xi) + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi) \right\}. \end{aligned}$$

Then by Proposition 5.1, we know that for $\xi = \xi_1$

$$(5.4) \quad 2A\phi AY = \alpha(A\phi + \phi A)Y - \phi Y - \phi_1Y + 2\{\eta_2(Y)\xi_3 - \eta_3(Y)\xi_2\}.$$

So it follows that $2A\phi AY = \alpha(A\phi + \phi A)Y - \phi Y - \phi_1Y$ for any $Y \in \mathcal{Q}$, where \mathcal{Q} denotes a maximal quaternionic subbundle of $T_x M$, $x \in M$. Then this gives the following

$$\begin{aligned} 2\phi A^2\phi AY &= 2\phi A(A\phi AY) = \phi A\{\alpha(A\phi + \phi A)Y - \phi Y - \phi_1Y\} \\ &= \alpha\phi A(A\phi + \phi A)Y - \phi A\phi Y - \phi A\phi_1Y. \end{aligned}$$

From this, together with (5.4), it follows that

$$(5.5) \quad \begin{aligned} & -5AY + 5\alpha\eta(Y)\xi + 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 + 2hA^2Y - 2A^3Y \\ & = 3\phi\phi_1AY + 2(h\alpha - \alpha^2)\alpha\eta(Y)\xi - h\phi\{\alpha(A\phi + \phi A)Y - \phi Y - \phi_1Y\} \\ & \quad + \alpha\phi A(A\phi + \phi A)Y - \phi A\phi Y - \phi A\phi_1Y \end{aligned}$$

Then (5.5) can be written as follows:

$$(5.6) \quad \begin{aligned} & 3\phi\phi_1AY - 2hA^2Y + 2A^3Y + 5AY = h\alpha\phi(A\phi + \phi A)Y + hY - h\phi\phi_1Y \\ & + 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 - \alpha\phi A(A\phi + \phi A)Y \\ & + \phi A\phi Y + \phi A\phi_1Y. \end{aligned}$$

On the other hand, the commuting Ricci tensor $\phi S = S\phi$ gives for any $Y \in \mathcal{Q}$ and $\xi = \xi_1 \in \mathcal{Q}^\perp$

$$\begin{aligned} & -\frac{1}{2}[(4m+7)\phi Y + \phi_1\phi^2Y] + hA\phi Y - A^2\phi Y \\ & = -\frac{1}{2}[(4m+7)\phi Y + \phi\phi_1\phi Y] = h\phi AY - \phi A^2Y \end{aligned}$$

Then for any Y belonging to the maximal quaternionic subbundle \mathcal{Q} , we know

$$hA\phi Y - A^2\phi Y = h\phi AY - \phi A^2Y.$$

From this, by replacing Y by ϕY for $Y \in \mathcal{Q}$ and applying A to the obtained equation, we have

$$(5.7) \quad hA^2Y - A^3Y = -hA\phi A\phi Y + A\phi A^2\phi Y.$$

On the other hand, by using (5.4) for any $\phi Y \in \mathcal{Q}$, we know

$$2A\phi A\phi Y = \alpha(A\phi + \phi A)\phi Y + Y - \phi_1\phi Y$$

and

$$\begin{aligned} 2A\phi A^2\phi Y &= \alpha(A\phi + \phi A)A\phi Y - \phi A\phi Y - \phi_1 A\phi Y \\ &\quad + 2\{\eta_2(A\phi Y)\xi_3 - \eta_3(A\phi Y)\xi_3\}. \end{aligned}$$

Then substituting these formulas into (5.7), we have

$$\begin{aligned} -2hA^2Y + 2A^3Y &= 2hA\phi A\phi Y - 2A\phi A^2\phi Y \\ &= h\alpha(A\phi + \phi A)\phi Y + hY - h\phi_1\phi Y \\ &\quad - \alpha(A\phi + \phi A)A\phi Y + \phi A\phi Y \\ &\quad - 2\{\eta_2(A\phi Y)\xi_3 - \eta_3(A\phi Y)\xi_2\}. \end{aligned}$$

From this, together with (5.6), it follows that

$$\begin{aligned} (5.8) \quad &3\phi_1\phi AY + 5AY + h\alpha(A\phi + \phi A)\phi Y + hY - h\phi_1\phi Y \\ &\quad - \alpha(A\phi + \phi A)A\phi Y + \phi A\phi Y + \phi_1 A\phi Y \\ &\quad - 2\{\eta_2(A\phi Y)\xi_3 - \eta_3(A\phi Y)\xi_2\} \\ &= \alpha h(A\phi + \phi A) + hY - h\phi_1\phi Y + 6\eta_2(AY)\xi_2 + 6\eta_3(AY)\xi_3 \\ &\quad - \alpha\phi A(A\phi + \phi A)Y + \phi A\phi Y + \phi A\phi_1 Y. \end{aligned}$$

Then it can be rearranged as follows:

$$\begin{aligned} (5.9) \quad &6\phi\phi_1 AY + 10AY - 2h\alpha AY - 2\alpha A\phi A\phi Y + 2\phi_1 A\phi Y \\ &\quad - 4\{\eta_2(A\phi Y)\xi_3 - \eta_3(A\phi Y)\xi_2\} \\ &= -2\alpha hAY + 12\eta_2(AY)\xi_2 + 12\eta_3(AY)\xi_3 \\ &\quad - 2\alpha\phi A\phi AY + 2\phi A\phi_1 Y. \end{aligned}$$

On the other hand, from (5.4) we calculate the following formulas

$$2\alpha A\phi A\phi Y = \alpha^2(A\phi + \phi A)\phi Y + \alpha Y - \alpha\phi_1\phi Y$$

and

$$2\alpha\phi A\phi AY = \alpha^2\phi(A\phi + \phi A)Y + \alpha Y - \alpha\phi\phi_1 Y.$$

Substituting these formulas into (5.9), we have

$$\begin{aligned} (5.10) \quad &6\phi\phi_1 AY + 10AY - 2h\alpha AY - \alpha^2(A\phi + \phi A)\phi Y - \alpha Y + \alpha\phi_1\phi Y + 2\phi_1 A\phi Y \\ &\quad - 4\{\eta_2(A\phi Y)\xi_3 - \eta_3(A\phi Y)\xi_2\} \\ &= -2\alpha hAY + 12\eta_2(AY)\xi_2 + 12\eta_3(AY)\xi_3 - \alpha^2\phi(A\phi + \phi A)Y \\ &\quad - \alpha Y + \alpha\phi\phi_1 Y + 2\phi A\phi_1 Y. \end{aligned}$$

Thus it can be rearranged for any $Y \in \mathcal{Q}$:

$$\begin{aligned} (5.11) \quad &6\phi\phi_1 AY + 10AY + 2\phi_1 A\phi Y - 4\{\eta_2(A\phi Y)\xi_3 - \eta_3(A\phi Y)\xi_2\} \\ &= 12\eta_2(AY)\xi_2 + 12\eta_3(AY)\xi_3 + \alpha^2 AY + 2\phi A\phi_1 Y. \end{aligned}$$

From this, if we take an inner product with ξ_2 , it follows that

$$6\eta_2(AY) + 10\eta_2(AY) - 2\eta_3(A\phi Y) + 4\eta_3(A\phi Y) = 12\eta_2(AY) + 2\eta_3(A\phi_1 Y).$$

Then it can be arranged by

$$(5.12) \quad \eta_3(A\phi Y) = -2\eta_2(AY) + \eta_3(A\phi_1 Y).$$

Similarly, let us take an inner product (5.11) by ξ_3 . Then we have

$$6\eta_3(AY) + 10\eta_3(AY) + 2\eta_2(A\phi Y) - 4\eta_2(A\phi Y) = 12\eta_3(AY) - 2\eta_2(A\phi_1 Y).$$

Then it can be arranged by

$$(5.13) \quad \eta_2(A\phi Y) = 2\eta_3(AY) + \eta_2(A\phi_1 Y)$$

for any $Y \in \mathcal{Q}$. Note that on a maximal quaternionic subbundle \mathcal{Q} we know that $(\phi\phi_1)^2 = I$, that is, $\phi\phi_1\phi\phi_1 X = X$ for any $X \in T_x M$, $x \in M$ and $\text{tr}(\phi\phi_1) = 0$. By virtue of these facts, we can decompose a maximal quaternionic subbundle \mathcal{Q} in such two eigenspaces as follows:

$$E_{+1} = \{X \in \mathcal{Q} | \phi\phi_1 X = X\}$$

and

$$E_{-1} = \{X \in \mathcal{Q} | \phi\phi_1 X = -X\},$$

where $\mathcal{Q} = E_{+1} \oplus E_{-1}$. Then we know that

$$X \in E_{+1} \text{ iff } \phi\phi_1 X = X \text{ iff } \phi_1 X = -\phi X$$

and

$$X \in E_{-1} \text{ iff } \phi\phi_1 X = -X \text{ iff } \phi X = \phi_1 X.$$

First let us consider on the subbundle $E_{-1} = \{Y \in \mathcal{Q} | \phi Y = \phi_1 Y\}$. Then (5.12) gives $\eta_2(AY) = 0$ for any $Y \in E_{-1}$. Moreover, (5.13) implies $\eta_3(AY) = 0$ for any $Y \in E_{-1}$.

Now let us consider a subbundle $E_{+1} = \{X \in \mathcal{Q} | \phi X = -\phi_1 X\}$ in the maximal subbundle \mathcal{Q} . Then (5.12) and (5.13) respectively gives the following

$$(5.14) \quad \eta_3(A\phi Y) = \eta_2(AY) \text{ and } \eta_2(A\phi Y) = -\eta_3(AY).$$

From this, together with (5.11), we have

$$(5.15) \quad 3\phi\phi_1 AY + 5AY + \phi_1 A\phi Y - 4\{\eta_2(AY)\xi_2 + \eta_3(AY)\xi_3\} = \phi A\phi_1 Y.$$

Now let us consider eigenvectors $Y, \phi Y \in E_{+1}$ such that $\phi Y = -\phi_1 Y$. Then we may put

$$\begin{aligned} AY &= \lambda Y + \sum_{\nu=1}^3 \eta_\nu(AY)\xi_\nu \\ A\phi Y &= \bar{\lambda}\phi Y + \sum_{\nu=1}^3 \eta_\nu(A\phi Y)\xi_\nu \end{aligned}$$

and

$$\phi AY = \lambda\phi Y + \sum_{\nu=1}^3 \eta_\nu(AY)\phi\xi_\nu.$$

From these formulas it follows that for any $Y \in E_{+1}$ satisfying $\phi Y = -\phi_1 Y$

$$\begin{aligned} \phi\phi_1 AY &= \phi\{\lambda\phi_1 Y + \sum_{\nu=1}^3 \eta_\nu(AY)\phi_1\xi_\nu\} = \lambda Y + \sum_{\nu=1}^3 \eta_\nu(AY)\phi\phi_1\xi_\nu, \\ \phi_1\phi A\phi Y &= \phi_1\{\bar{\lambda}\phi Y + \sum_{\nu=1}^3 \eta_\nu(A\phi Y)\xi_\nu\} = \bar{\lambda}Y + \sum_{\nu=1}^3 \eta_\nu(A\phi Y)\phi_1\xi_\nu, \end{aligned}$$

and

$$\phi A \phi Y = -\{\bar{\lambda}Y - \sum_{\nu=1}^3 \eta_{\nu}(A \phi Y) \phi \xi_{\nu}\} = -\phi A \phi_1 Y.$$

From these formulas, (5.15) gives the following

$$\begin{aligned} (5.16) \quad & 3\{\lambda Y + \sum_{\nu=1}^3 \eta_{\nu}(AY) \phi \phi_1 \xi_{\nu}\} + 5\{\lambda Y + \sum_{\nu=1}^3 \eta_{\nu}(AY) \xi_{\nu}\} \\ & + \bar{\lambda}Y + \sum_{\nu=1}^3 \eta_{\nu}(A \phi Y) \phi_1 \xi_{\nu} \\ & = 4\{\eta_2(AY) \xi_2 + \eta_3(AY) \xi_3\} + \{\bar{\lambda}Y - \sum_{\nu=1}^3 \eta_{\nu}(A \phi Y) \phi \xi_{\nu}\} \end{aligned}$$

which gives $\lambda = 0$. Then this implies the following

$$(5.17) \quad AY = \sum_{\nu=1}^3 \eta_{\nu}(AY) \xi_{\nu} = g(A \xi_2, Y) \xi_2 + g(A \xi_3, Y) \xi_3.$$

Then for any $\phi Y \in E_{+1}$ we know that

$$\begin{aligned} A \phi Y &= g(A \xi_2, \phi Y) \xi_2 + g(A \xi_3, \phi Y) \xi_3 \\ &= -\eta_3(AY) \xi_2 + \eta_2(AY) \xi_3 \\ \phi A \phi Y &= g(A \xi_2, \phi Y) \phi \xi_2 + g(A \xi_3, \phi Y) \phi \xi_3 \\ &= -g(A \xi_2, \phi Y) \xi_3 + g(A \xi_3, \phi Y) \xi_2 \\ \phi_1 A \phi Y &= g(A \xi_2, \phi Y) \xi_3 - g(A \xi_3, \phi Y) \xi_2. \end{aligned}$$

Substituting these fomulas into (5.15) and using $\phi Y = -\phi_1 Y$ for $Y \in E_{+1}$, we have

$$\begin{aligned} (5.18) \quad & 3\phi \phi_1 AY = -5AY - \phi_1 A \phi Y + 4\{\eta_2(AY) \xi_2 + \eta_3(AY) \xi_3\} - \phi A \phi Y \\ & = -5\{\eta_2(AY) \xi_2 + \eta_3(AY) \xi_3\} + 4\{\eta_2(AY) \xi_2 + \eta_3(AY) \xi_3\} \\ & = -\{\eta_2(AY) \xi_2 + \eta_3(AY) \xi_3\} \\ & = 3\phi_1 \phi AY. \end{aligned}$$

Then by applying ϕ_1 to the second equality of the above equation, we know that

$$3\phi AY = \eta_2(AY) \xi_3 - \eta_3(AY) \xi_2$$

for any $Y \in E_{+1}$, which gives that

$$-3AY = \eta_2(AY) \phi \xi_3 - \eta_3(AY) \phi \xi_2 = \eta_2(AY) \xi_2 + \eta_3(AY) \xi_3.$$

From this, by applying ξ_2 and ξ_3 , we get the following respectively

$$\eta_2(AY) = 0 \quad \text{and} \quad \eta_3(AY) = 0$$

for any $Y \in E_{+1}$. Summing up the case in the subbundle E_{-1} and the fact that M is Hopf we proved that $g(AQ, Q^{\perp}) = 0$, that is the maximal quaternionic subbundle Q is invariant under the shape operator. Then from such a view point, by Theorem A in the introduction we conclude that M is locally congruent to one of real hypersurfaces of type (A), (B), (C₁), (C₂) and (D). Among them we know that the Reeb vector field ξ of hypersurfaces of type (A) and (C₁) belongs to the

quaternionic maximal subbundle \mathcal{Q} , that is $JN \in \mathfrak{J}N$. This gives a complete proof of our main theorem. \square

On the other hand, the Reeb flow of these type hypersurfaces mentioned in our theorem is isometric as in Theorem B. That is, the shape operator commutes with the structure tensor, that is $A\phi = \phi A$. So by using this property, conversely, let us check whether or not these hypersurfaces satisfy the Ricci commuting. In order to do this, let us introduce the following Proposition due to Berndt and Suh [3] :

Proposition B. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Assume that the maximal complex subbundle \mathcal{C} of TM and the maximal quaternionic subbundle \mathcal{Q} of TM are both invariant under the shape operator of M . If $JN \in \mathfrak{J}N$, then one the following statements hold:*

(A) *M has exactly four distinct constant principal curvatures*

$$\alpha = 2 \coth(2r) , \beta = \coth(r) , \lambda_1 = \tanh(r) , \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C} , T_\beta = \mathcal{C} \ominus \mathcal{Q} , T_{\lambda_1} = E_{-1} , T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to \mathfrak{J}).

(C₁) *M has exactly three distinct constant principal curvatures*

$$\alpha = 2 , \beta = 1 , \lambda = 0$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus \mathcal{C} , T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1} , T_\lambda = E_{+1}.$$

Now by using above proposition and the isometric Reeb flow, let us check real hypersurfaces of type (A) and (C₂) satisfy $S\phi = \phi S$. The from $A\phi = \phi A$ the Ricci commuting gives the following

$$\begin{aligned} (5.19) \quad & -3 \sum_{\nu=1}^3 \eta_\nu(\phi Y) \xi_\nu + \sum_{\nu=1}^3 \{ \eta_\nu(\xi) \phi_\nu \phi^2 Y - \eta(\phi_\nu \phi Y) \phi_\nu \xi \} \\ & = -3 \sum_{\nu=1}^3 \eta_\nu(Y) \phi \xi_\nu + \sum_{\nu=1}^3 \{ \eta_\nu(\xi) \phi \phi_\nu \phi Y - \eta(\phi_\nu Y) \phi \phi_\nu \xi \}. \end{aligned}$$

Now let us check it for real hypersurfaces of type (A) and of type (C₁) in Proposition B as follows:

Case 1) Put $Y = \xi = \xi_1$. Then the both sides equal to each other.

Case 2) Put $Y = \xi_2, \xi_3$. For $Y = \xi_2$ in (5.19) we have

$$\begin{aligned} & -3\eta_3(\phi \xi_2 \xi_3) + \phi_1 \phi^2 \xi_2 - \eta(\phi_2 \phi \xi_2) \phi_2 \xi \\ & = -3\phi \xi_2 + \phi \phi_1 \phi \xi_2 - \eta(\phi_3 \xi_2) \phi \phi_3 \xi \\ & = \xi_3. \end{aligned}$$

Also the both sides hold for $Y = \xi_3$.

Case 3) Put $Y \in E_{-1} \oplus E_{+1}$. First let us say $Y \in E_{-1}$ such that $\phi Y = -\phi_1 Y$. The left side of (5.19) becomes

$$L = \sum_{\nu=1}^3 \eta_\nu(\xi) \phi_\nu \phi \phi^2 Y = \phi_1 \phi^2 Y = -\phi_1 Y$$

and the Right side is given by

$$R = \phi \phi_1 \phi Y = \phi_1 \phi \phi Y = \phi_1 \phi^2 Y = -\phi_1 Y.$$

Also the both sides equal to each other for $Y \in E_{+1}$.

Remark 5.2. In the paper [18] we have given a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and have proved that they are locally congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Also in [18] we have proved that they have isometric Reeb flows.

Remark 5.3. In the paper [16] we have given a complete classification of pseudo-Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$ and have found that there does not exist any Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Remark 5.4. Our main theorem in the introduction will give a contribution to the study of Lie invariant problems for real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ or to the classification problem of pseudo-Einstein real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$. Moreover, based on the classification of the isometric Reeb flow in [4], it will give a contribution to the study of real hypersurfaces in complex quadric Q^m with commuting Ricci tensor.

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